Transformation Formulas of p-adic Hypergeometric Functions

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Dwork's p-adic hypergeometric functions $\mathscr{F}_{\underline{a}}^{Dw}(t)$

Let $s \geq 1$ be an integer.

For a s-tuple $\underline{a} = (a_1, ..., a_s) \in \mathbb{Z}_p^s$ of p-adic integers, let

$$F_{\underline{a}}(t) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} \cdots \frac{(a_s)_k}{k!} t^k$$

be the p-adic hypergeometric power series where

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \quad \textit{when} \quad k \geq 1 \textit{ and } \quad (\alpha)_0 = 1.$$

Let $a' := (a+l)/p$ where $l \in \{0, 1, ..., p-1\}$ is the unique integer such that $a + l \equiv 0 \mod p$ (a' is called the **Dwork prime** of a).

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Example

Let $p = 5$. We have

$$1' = 1, \ (\frac{1}{2})' = \frac{\frac{1}{2}+2}{5} = \frac{1}{2} \text{ and } (\frac{1}{3})' = \frac{\frac{1}{3}+3}{5} = \frac{2}{3}.$$

Similarly, the i-th Dwork prime $a^{(i)}$ is defined by $a^{(i)} = (a^{(i-1)})'$ and $a^{(0)} = a$.

Put $\underline{a'} = (a'_1, \cdots, a'_s)$.

Definition

The Dwork's p-adic hypergeometric function is defined to be

$$\mathscr{F}^{Dw}_{\underline{a}}(t) = F_{\underline{a}}(t)/F_{\underline{a'}}(t^p).$$

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Example

Taking $a_1 = \cdots = a_s = 1$, *we have*

$$\mathscr{F}_{\underline{a}}^{Dw}(t) = F_{1,\cdots,1}(t)/F_{1,\cdots,1}(t^p) = \frac{\sum t^n}{\sum t^{pn}} = \frac{\frac{1}{1-t}}{\frac{1}{1-t^p}} = 1 + t + \cdots + t^{p-1}.$$

Congruence relations

Dwork's p-adic hypergeometric functions satisfy congruence relations.

Theorem (Dwork)

We have

$$\mathscr{F}_{\underline{a}}^{Dw}(t) \equiv \frac{F_{\underline{a}}(t)_{<p^n}}{[F_{\underline{a}'}(t^p)]_{<p^n}} \mod p^n\mathbb{Z}_p[[t]]$$

where $f(t)_{<m} := \sum_{n<m} A_n t^n$ *is the truncated polynomial for a power series* $f(t) = \sum_{n=0}^{\infty} A_n t^n$.

Example

Taking $a_1 = \cdots = a_s = 1$, *we have*

$$\frac{F_{1,\cdots,1}(t)_{<p}}{F_{1,\cdots,1}(t^p)_{<p}} = \frac{(1+t+t^2+\cdots)_{<p}}{(1+t^p+t^{2p}+\cdots)_{<p}}$$
$$= 1+t+\cdots+t^{p-1} \equiv \mathscr{F}^{Dw}_{1,\cdots,1}(t) \mod p$$

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$$= 1+t+\cdots+t^{p-1} \equiv \mathscr{F}^{Dw}_{1,\cdots,1}(t) \mod p$$

and

$$\frac{F_{1,\cdots,1}(t)_{<p^2}}{F_{1,\cdots,1}(t^p)_{<p^2}} = \frac{1+t+\cdots+t^{p^2-1}}{1+t^p+t^{2p}+\cdots+t^{p(p-1)}}$$
$$= \frac{\frac{t^{p^2}-1}{t-1}}{\frac{t^{p^2}-1}{t^p-1}} = 1+t+\cdots+t^{p-1} \equiv \mathscr{F}^{Dw}_{1,\cdots,1}(t) \mod p^2.$$

From Dwork's congruence relation, we have

$$\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t) \in \mathbb{Z}_p\langle t, f(t)^{-1}\rangle := \varprojlim_n (\mathbb{Z}_p/p^n\mathbb{Z}_p[t, f(t)^{-1}])$$

where

$$f(t) := \prod_{i=0}^{N} F_{a_1^{(i)},\cdots,a_s^{(i)}}(t)_{<p}$$

and N is an integer such that $\underline{a}^{(N)} = \underline{a}$.

Special values of $\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t)$

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ and $\mathcal{O}_{\mathbb{C}_p} := \{|x|_p \leq 1\}$ the valuation ring and $m := \{|x|_p < 1\}$ the maximal ideal.

Special values of $\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t)$

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ and $\mathcal{O}_{\mathbb{C}_p} := \{|x|_p \leq 1\}$ the valuation ring and $m := \{|x|_p < 1\}$ the maximal ideal.

For $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ satisfying

$$F_{\underline{a}'}(\alpha)_{<p^n} \not\equiv 0 \mod m, \forall n,$$

the special value $\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(\alpha)$ is defined to be

$$\lim_{n\to\infty} \left(\frac{F_{\underline{a}}(t)_{<p^n}}{F_{\underline{a}'}(t^p)_{<p^n}} \Big|_{t=\alpha} \right).$$

Geometric aspect of $\mathscr{F}^{Dw}_{a_1,\cdots,a_s}(t)$

Dwork showed a geometric aspect of his p-adic hypergeometric functions by his unit root formula.

Theorem (unit root formula)

For a smooth ordinary elliptic curve

$$E_\alpha : y^2 = x(1-x)(1-\alpha x)$$

over $\mathbb{F}_p$, the unit root ϵ_p of E_α satisfies

$$\epsilon_p = (-1)^{\frac{p-1}{2}} \mathscr{F}^{Dw}_{\frac{1}{2},\frac{1}{2}}(\widehat{\alpha})$$

where $\widehat{\alpha} \in \mathbb{Z}_p^\times$ is the Teichmüller lift of $\alpha \in \mathbb{F}_p^\times$.

Unit root is the root of $x^2 - a_p x + p$ which is unit ($a_p := \#E(\mathbb{F}_p) - p - 1$).

p-adic hypergeometric functions of logarithmic type $\mathscr{F}^{(\sigma)}_{a_1,\cdots,a_s}(t)$

Let $W = W(\overline{\mathbb{F}}_p)$ denote the Witt ring, and K=Frac$W$ its fractional field.

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Let $W = W(\overline{\mathbb{F}}_p)$ denote the Witt ring, and K=Frac$W$ its fractional field.

Let $\sigma : W[[t]] \to W[[t]]$ be a p-th Frobenius given by $\sigma(t) = ct^p$ with $c \in 1 + pW$:

$$\left(\sum_i a_i t^i\right)^{\sigma} = \sum_i a_i^F c^i t^{ip}$$

where $F : W \to W$ is the Frobenius on W.

Define p-adic digamma function

$$\psi_p(z) := -\gamma_p + \lim_{n \in \mathbb{Z}_{>0}, n \to z} \sum_{1 \le k < n, p \nmid k} \frac{1}{k}$$

where γ_p is

$$-\lim_{s \to \infty} \frac{1}{p^s} \sum_{0 \le j < p^s, p \nmid j} \log(j), \quad (\log = \text{Iwasawa log})$$

Definition (M. Asakura)

We define

$$\mathscr{F}_{\underline{a}}^{(\sigma)}(t) = \frac{G_{\underline{a}}(t)}{F_{\underline{a}}(t)} = \frac{1}{F_{\underline{a}}(t)}\Big[\psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p - p^{-1}\log(c) + \int_0^t (F_{\underline{a}}(t) - F_{\underline{a}'}(t^\sigma))\frac{dt}{t}\Big]$$

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Here we think $\int_0^t (-)\frac{dt}{t}$ to be a operator such that

$$\int_0^t t^\alpha \frac{dt}{t} = \frac{t^\alpha}{\alpha}, \quad \alpha \neq 0.$$

Write $F_{\underline{a}}(t) = \sum A_k t^k$, $F_{\underline{a}'}(t) = \sum A_k^{(1)} t^k$ and $G_{\underline{a}}(t) = \sum B_k t^k$.

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Then for $k \in \mathbb{Z}_{\geq 0}$, we have

$$B_0 = \psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p - p^{-1}\log(c), \quad B_k = \frac{A_k - c^{k/p} A_{k/p}^{(1)}}{k}$$

where $A_{\frac{m}{p}}^{(1)} = 0$ if $m \not\equiv 0 \mod p$ or $m < 0$.

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where $A_{\frac{m}{p}}^{(1)} = 0$ if $m \not\equiv 0 \mod p$ or $m < 0$.

These p-adic hypergeometric functions of logarithmic type also satisfy congruence relations.

Theorem (M. Asakura)

Suppose that $a_i \notin \mathbb{Z}_{\leq 0}$ for all i. If $c \in 1 + 2pW$, then for all $n \geq 1$

$$\mathscr{F}_{\underline{a}}^{(\sigma)}(t) \equiv \frac{G_{\underline{a}}(t)_{<p^n}}{F_{\underline{a}}(t)_{<p^n}} \mod p^n W[[t]].$$

If $p = 2$ and $c \in 1 + 2W$, then the above formula holds for modulo p^{n-1}.

Geometric aspect of $\mathscr{F}^{(\sigma)}_{a_1,\cdots,a_s}(t)$

Theorem (Asakura)

Suppose that $p > 2$ is prime to NM. Let $F : z^N + w^M = 1$ be the Fermat curve over W. Let

$$\mathrm{reg}_{\mathrm{syn}} : K_2(F) \otimes \mathbb{Q} \to H^2_{\mathrm{syn}}(F, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(F/K)$$

be the syntomic regulator map and let $A(i,j) \in K$ be defined by

$$\mathrm{reg}_{\mathrm{syn}}(\{1-z, 1-w\}) = \sum_{(i,j)\in I} A^{(i,j)} M^{-1} z^{i-1} w^{j-M} dz.$$

Suppose that $(i,j) \in I$ satisfies $(i)\frac{i}{N} + \frac{j}{M} < 1, \quad (ii)F_{\frac{i}{N},\frac{j}{M}}(1)_{<p^n} \equiv 0$ mod $p, \forall n \geq 1$. Then we have

$$A^{(i,j)} = \mathscr{F}^{(\sigma)}_{\frac{i}{N},\frac{j}{M}}(1) \quad \textit{where} \quad \sigma(t) = t^p.$$

p-adic Hypergeometric Functions $\widehat{\mathscr{F}}^{(\sigma)}_{a,\dots,a}(t)$

For $a \in \mathbb{Z}_p$, we write

$$F_{a,\dots,a}(t) = \sum_{k=0}^{\infty} \left(\frac{(a)_k}{k!} \right)^s t^k, \quad F_{a',\dots,a'}(t) = \sum_{k=0}^{\infty} \left(\frac{(a')_k}{k!} \right)^s t^k.$$

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Put

$$q := \begin{cases} 4 & p = 2 \\ p & p \geq 3. \end{cases}$$

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Put

$$q := \begin{cases} 4 & p = 2 \\ p & p \geq 3. \end{cases}$$

Let $l' \in \{0, 1, \dots, q-1\}$ be the unique integer such that $a + l' \equiv 0 \mod q$.

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Put

$$q := \begin{cases} 4 & p = 2 \\ p & p \geq 3. \end{cases}$$

Let $l' \in \{0, 1, \dots, q-1\}$ be the unique integer such that $a + l' \equiv 0 \mod q$. Put

$$e := l' - \lfloor \frac{l'}{p} \rfloor.$$

Definition

We define

$$\widehat{G}^{(\sigma)}_{a,\ldots,a}(t) := t^{-a}\int_0^t (t^a F_{a,\ldots,a}(t) - (-1)^{se}[t^{a'} F_{a',\ldots,a'}(t)]^{\sigma})\frac{dt}{t}$$
$$= \sum_{k=0}^{\infty} \widehat{B}_k t^k \quad \textit{for } a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}.$$

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Definition

We define

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\dots,a}(t) := \frac{\widehat{G}^{(\sigma)}_{a,\dots,a}(t)}{F_{a,\dots,a}(t)}, \quad a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\le 0}.$$

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Definition

We define

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\dots,a}(t) := \frac{\widehat{G}^{(\sigma)}_{a,\dots,a}(t)}{F_{a,\dots,a}(t)}, \quad a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}.$$

Write $F_{a,\dots,a}(t) = \sum A_k t^k$ and $F_{a',\dots,a'}(t) = \sum A^{(1)}_k t^k$. Then we have

$$\widehat{B}_k = \frac{1}{k+a}\left(A_k - (-1)^{se}(A^{(1)}_{\frac{k-l}{p}})c^{\frac{k+a}{p}}\right).$$

We proved $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$ also satisty congruence relations in [W].

Theorem (Wang)

Let $a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}$ and suppose $c \in 1 + qW$. Then

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) \equiv \frac{\widehat{G}^{(\sigma)}_{a,\ldots,a}(t)_{<p^n}}{F_{a,\ldots,a}(t)_{<p^n}} \mod p^n W[[t]]$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Corollary

If $a^{(r)} = a$ *for some* $r > 0$ *where* $a^{(r)}$ *is the* r*-th Dwork's prime* $a^{(r)} = (a^{(r-1)})'$ *and* $a^{(0)} = a$.
Then

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) \in W\langle t, t^{-1}, h(t)^{-1}\rangle, \quad h(t) := \prod_{i=0}^{r-1} F_{a^{(i)},\ldots,a^{(i)}}(t)_{<p}$$

is a convergent function, where

$$W\langle t, t^{-1}, h(t)^{-1}\rangle := \varprojlim_{n}(W/p^n[t, t^{-1}, h(t)^{-1}])$$

.

Sketch of Proof

- Step I:Reduction to the case $c = 1$.
  We can prove if the congruence relations is true for $c = 1$, then it is true for any other c.

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- Step II: Prove the following lemma for $c = 1$.

Lemma

For $k, k' \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$, we have

$$k \equiv k' \mod p^n \Rightarrow \frac{\widehat{B}_k}{A_k} \equiv \frac{\widehat{B}_{k'}}{A_{k'}} \mod p^n.$$

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$$k \equiv k' \mod p^n \Rightarrow \frac{\widehat{B}_k}{A_k} \equiv \frac{\widehat{B}_{k'}}{A_{k'}} \mod p^n.$$

- Step III: Prove the congruence relation for $c = 1$.

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Transformation Formulas of p-adic Hypergeometric Functions

Conjecture (Transformation Formulas of p-adic Hypergeometric Functions)

Let $\sigma(t) = ct^p$ *and* $\widehat{\sigma}(t) = c^{-1}t^p$. *Suppose* $a^{(r)} = a$ *for some* $r > 0$. *Then*

$$\mathscr{F}^{(\sigma)}_{a,\ldots,a}(t) = -\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\ldots,a}(t^{-1})$$

in the ring $W\langle t, t^{-1}, h(t)^{-1}\rangle$ *where* $h(t) := \prod_{i=0}^{r-1} F_{a^{(i)},\ldots,a^{(i)}}(t)_{<p}$.

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in the ring $W\langle t, t^{-1}, h(t)^{-1}\rangle$ *where* $h(t) := \prod_{i=0}^{r-1} F_{a^{(i)},\ldots,a^{(i)}}(t)_{<p}$.

Remark

There is an involution

$$\omega : W\langle t, t^{-1}, h(t)^{-1}\rangle \longrightarrow W\langle t, t^{-1}, h(t)^{-1}\rangle, \quad \omega(f(t)) = f(t^{-1}).$$

By congruence relations of $\mathscr{F}_a^{(\sigma)}(t)$ and $\widehat{\mathscr{F}_a}^{(\widehat{\sigma})}(t)$, the conjecture of transformation formulas is equivalent to the statement

$$\frac{G_a^{(\sigma)}(t)_{<p^n}}{F_a(t)_{<p^n}} \equiv - \left. \frac{\widehat{G}_a^{(\widehat{\sigma})}(t)_{<p^n}}{F_a(t)_{<p^n}} \right|_{t^{-1}} \mod p^n W[[t]]$$

for all $n \in \mathbb{Z}_{\geq 0}$.

By congruence relations of $\mathscr{F}_a^{(\sigma)}(t)$ and $\widehat{\mathscr{F}_a}^{(\widehat{\sigma})}(t)$, the conjecture of transformation formulas is equivalent to the statement

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for all $n \in \mathbb{Z}_{\geq 0}$.

So it suffices to show that

$$\sum_{\substack{i+j=m \\ 0\leq i,j\leq p^n-1}} B_i A_{p^n-j-1} + \widehat{B}_{p^n-j-1} A_i \equiv 0 \mod p^n.$$

for any m with $0 \leq m \leq 2(p^n - 1)$.

Transformation Formulas of p-adic Hypergeometric Functions

Conjecture (Transformation Formulas of Dwork's p-adic Hypergeometric Functions)

Let l is the unique integer in $\{0, 1, \cdots, p-1\}$ such that $a + l \equiv 0 \mod p$. Then for odd prime p, we have

$$\mathscr{F}^{\mathrm{Dw}}_{a,\cdots,a}(t) = ((-1)^s t)^l \mathscr{F}^{\mathrm{Dw}}_{a,\cdots,a}(t^{-1}).$$

For $p = 2$, we have

$$\mathscr{F}^{\mathrm{Dw}}_{a,\cdots,a}(t) = \pm((-1)^s t)^l \mathscr{F}^{\mathrm{Dw}}_{a,\cdots,a}(t^{-1})$$

where $\pm$ depends on a and s.

Transformation Formulas – Case $s = 1$

Theorem

Let p be an odd prime and $a \in \mathbb{Z}_p$, then we have

$$\mathscr{F}_a^{\mathrm{Dw}}(t) = (-t)^l \mathscr{F}_a^{\mathrm{Dw}}(t^{-1}),$$

where l is the unique integer in $\{0, 1, \cdots, p-1\}$ such that $a + l \equiv 0 \mod p$.

For $p = 2$, we have

$$\mathscr{F}_a^{\mathrm{Dw}}(t) = \pm(-t)^l \mathscr{F}_a^{\mathrm{Dw}}(t^{-1}),$$

($+ : a' \equiv 0 \mod 2$; $- : a' \equiv 1 \mod 2$)

Proof.

By congruence relation of Dwork p-adic hypergeometric function, we have

$$\mathscr{F}_a^{\mathrm{Dw}}(t) \equiv \frac{F_a(t)_{<p^n}}{F_{a'}(t^p)_{<p^n}} = \frac{(1-t)^{-a}_{<p^n}}{(1-t^p)^{-a'}_{<p^n}} \equiv \frac{(1-t)^{-N}_{<p^n}}{(1-t^p)^{-N'}_{<p^n}} \mod p^n$$

for some $N \in \mathbb{Z}_{>0}$ and $a \equiv N \mod p$.

Proof.

By congruence relation of Dwork p-adic hypergeometric function, we have

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for some $N \in \mathbb{Z}_{>0}$ and $a \equiv N \mod p$.

Here we assume p is odd ($p = 2$ is similar); then we obtain

$$\mathscr{F}_a^{\mathrm{Dw}}(t^{-1}) \equiv \left.\frac{(1-t)^{-N}_{<p^n}}{(1-t^p)^{-N'}_{<p^n}}\right|_{t^{-1}} \mod p^n$$

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for some $N \in \mathbb{Z}_{>0}$ and $a \equiv N \mod p$.

Here we assume p is odd ($p = 2$ is similar); then we obtain

$$\begin{aligned} \mathscr{F}_a^{\mathrm{Dw}}(t^{-1}) &\equiv \left.\frac{(1-t)^{-N}_{<p^n}}{(1-t^p)^{-N'}_{<p^n}}\right|_{t^{-1}} \mod p^n \\ &\equiv \left.\frac{(1-t)^{-N}}{(1-t^p)^{-N'}}\right|_{t^{-1}} \end{aligned}$$

Proof.

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for some $N \in \mathbb{Z}_{>0}$ and $a \equiv N \mod p$.

Here we assume p is odd ($p = 2$ is similar); then we obtain

$$\begin{aligned}
\mathscr{F}_a^{\mathrm{Dw}}(t^{-1}) &\equiv \left.\frac{(1-t)^{-N}_{<p^n}}{(1-t^p)^{-N'}_{<p^n}}\right|_{t^{-1}} \mod p^n \\
&\equiv \left.\frac{(1-t)^{-N}}{(1-t^p)^{-N'}}\right|_{t^{-1}} \\
&= \frac{(-t)^{-pN'}}{(-t)^{-pN'}} \cdot \frac{(1-t^{-1})^{-N}}{(1-t^{-p})^{-N'}}
\end{aligned}$$

Proof.

By congruence relation of Dwork p-adic hypergeometric function, we have

$$\mathscr{F}_a^{\mathrm{Dw}}(t) \equiv \frac{F_a(t)_{<p^n}}{F_{a'}(t^p)_{<p^n}} = \frac{(1-t)^{-a}_{<p^n}}{(1-t^p)^{-a'}_{<p^n}} \equiv \frac{(1-t)^{-N}_{<p^n}}{(1-t^p)^{-N'}_{<p^n}} \mod p^n$$

for some $N \in \mathbb{Z}_{>0}$ and $a \equiv N \mod p$.

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&= \frac{(-t)^{-l}(1-t)^{-N}}{(1-t^p)^{-N'}}.
\end{aligned}$$

Theorem (Wang)

Transformation formulas of $\mathscr{F}^{Dw}_{a,\cdots,a}(t)$ *imply transformation formulas of* $\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t)$ *and* $\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\cdots,a}(t)$.

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(Sketch of proof): Let us define functions $f : \mathbb{Z}_{>0} \to W$ and $\widehat{f} : \mathbb{Z}_{>0} \to W$ by

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Since

$$k \equiv k' \mod p^n \implies \frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}}, \quad \frac{\widehat{B}_k}{A_k} \equiv \frac{\widehat{B}_{k'}}{A_{k'}} \mod p^n,$$

we can extend functions f and $\widehat{f}$ from $\mathbb{Z}_{>0}$ to $\mathbb{Z}_p$ denoted by β and $\widehat{\beta}$, respectively. We write the value of β(resp. $\widehat{\beta}$) at $\lambda \in \mathbb{Z}_p$ by β_λ(resp. $\widehat{\beta}_\lambda$).

Lemma

Let $\lambda \in \mathbb{Z}_p$ *and* $a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}$, *then we have*

$$\beta_\lambda + \widehat{\beta}_{-\lambda-a} = 0.$$

Lemma

Let $a \in \mathbb{Z}_p$, $m, k, d \in \mathbb{Z}_{\geq 0}$ *with* $0 \leq m \leq p^n - 1$ *and* $0 \leq d \leq n$.

If the transformation formula of Dwork p-adic hypergeometric function is true, then we have

$$\sum_{\substack{i \equiv k \mod p^{n-d} \\ 0 \leq i \leq m \\ i+j=m}} A_i A_{p^n-j-1} - \sum_{\substack{p^n-j'-1 \equiv -k-a \mod p^{n-d} \\ 0 \leq j' \leq m \\ i'+j'=m}} A_{i'} A_{p^n-j'-1} \equiv 0$$

modulo p^{d+1}.

Rewriting the summation

$$\sum_{\substack{i+j=m \\ 0 \le i,j \le p^n-1}} B_i A_{p^n-j-1} + \widehat{B}_{p^n-j-1} A_i$$

by the relation $\beta_k = B_k/A_k$ and $\widehat{\beta}_k = \widehat{B}_k/A_k$, we obtain

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Then using lemmas recursively, we obtain the result.

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Then using lemmas recursively, we obtain the result.

Corollary

Transformation formulas of ${\mathscr{F}_a}^{(\sigma)}(t)$ *and* $\widehat{\mathscr{F}_a}^{(\widehat{\sigma})}(t)$ *(i.e.,* $s = 1$*) are true.*

Transformation Formulas – Case: $s = 2$

Theorem (Wang)

Let $N \geq 2$ be an integer, prime $p > N$ and $a \in \frac{1}{N}\mathbb{Z}$ with $0 < a < 1$. Let $\sigma(t) = ct^p$ and $\widehat{\sigma}(t) = c^{-1}t^p$. Then

$$\mathscr{F}_{a,a}^{(\sigma)}(t) = -\widehat{\mathscr{F}}_{a,a}^{(\widehat{\sigma})}(t^{-1})$$

and

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Basic idea :

- Hypergeometric curves give p-adic hypergeometric functions.
- Automorphisms of hypergeometric curves give transformtation formulas of p-adic hypergeometric functions.

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